Open string fluctuations in $AdS_5\times S^5$ and operators with a large *R* charge

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A semiclassical string description is given for correlators of Wilson loops with local operators in $\mathcal{N}=4$ supersymmetric Yang-Mills theory in the regime when operators carry a parametrically large *R* charge. The operator product expansion coefficients of the circular Wilson loop in chiral primary operators are computed to all orders in the α' expansion in $AdS_5\times S^5$ string theory. The results agree with field-theory predictions.

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I. INTRODUCTION

String states with large quantum numbers are semiclassical and consequently are simple even in highly curved backgrounds. This fact has proven extremely useful in the context of the holographic duality between type IIB strings in $AdS_5\times S^5$ and $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory [1] and has led to an elegant string description of sectors of large spin or large *R* charge in SYM theory [2,3]. This description goes beyond the strong-coupling supergravity limit and can be compared to perturbative SYM theory at weak coupling [2–7]. The string states with a large *R* charge correspond to excitations of closed point-like strings that carry a large angular momentum in S^5 [3]. The purpose of the present paper is to study how these states couple to the open string sector, which is dual to Wilson loops in SYM theory [8–10]. Although open strings can also rotate in S^5, their angular momentum never becomes large [11]. To make contact with the limit of the large *R* charge, I will exploit the fact that any Wilson loop contains, as a composite operator, an admixture of states with any quantum number. It is natural to expect that the expansion coefficients of a Wilson loop in operators with a large *R* charge will have some kind of semiclassical string description.

The standard operator product expansion (OPE) of a Wilson loop [12,13] is defined as

$$W(C)=\langle W(C)\rangle \sum c_A R^{\Delta_A}\mathcal{O}_A(0), \tag{1}$$

where $\mathcal{O}_A(0)$ is an operator evaluated at the center of the loop, Δ_A is the conformal dimension of $\mathcal{O}_A(x)$, and *R* is the radius of the loop (*R* will be set to 1 in what follows). The dimensionless coefficients c_A depend on the shape of the contour *C* and, in the large-*N* limit, on the 't Hooft coupling $\lambda=g_{YM}^2N$. To start with, I will consider a specific case of the circular Wilson loop and of its expansion coefficients in the chiral primary operators. Then I will give some more general results in Sec. V. The operators of interest are

$$\mathcal{O}_J=\frac{(2\pi)^J}{\sqrt{J}\lambda^{J/2}}\mathrm{tr}Z^J, \tag{2}$$

where $Z=\Phi_1+i\Phi_2$. These operators have *R* charge *J* and dimension $\Delta=J$, and play an important role in the planar-wave limit of the AdS conformal field theory (CFT) correspondence [2]. The reason to choose the circular loop is twofold. First, there are a lot of explicit results for the circular loop on the string side of AdS/CFT duality [13–15]. Second, the invariance of the circular loop operator under certain superconformal transformations [16] partially protects it from gaining quantum corrections [17]. An expectation value of the circular loop [18,17] and some of its OPE coefficients [19] can be exactly calculated by resumming Feynman diagrams that survive supersymmetry cancellations. In particular, the weights with which operators (2) appear in the circular Wilson loop are known [19]:

$$c_J(\lambda)=\frac{1}{2}\sqrt{J\lambda}\frac{I_J(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}. \tag{3}$$

Here, $I_J(x)$ are modified Bessel functions. Some of their properties are listed in Appendix A. This expression is believed to be exact in the large-*N* limit.

In string theory, OPE coefficients of a Wilson loop measure an overlap of the boundary state associated with the loop with the closed string state associated with the local operator. The overlap can be computed in the supergravity approximation [13], which corresponds to taking the limit of large λ in Eq. (3). Further expansion in $1/\sqrt{\lambda}$ should be equivalent to the semiclassical (or α') expansion in string theory. The purpose of the present paper is to go beyond the supergravity limit directly in the $AdS_5\times S^5$ string theory and to extend the semiclassical approximation for OPE coefficients to all orders in α' for operators with large *R* charge *J*. Before going into details, let me illustrate on a simple example how large quantum numbers can modify a semiclassical expansion.

II. A TOY MODEL

Consider the integral

$$\langle \phi^n\rangle=\frac{\int D\phi\,\phi^n e^{-(1/\hbar)S(\phi)}}{\int D\phi\, e^{-(1/\hbar)S(\phi)}}, \tag{4}$$

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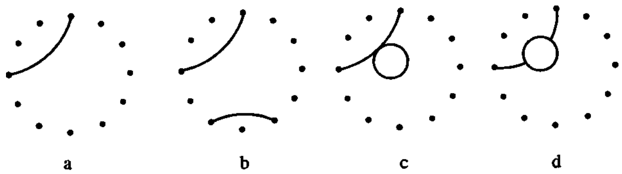

FIG. 1. The semiclassical expansion of $\langle\phi^n\rangle$ to order $O(\hbar^2)$. The first two diagrams are of order $\hbar n^2$ and $\hbar^2 n^4$, respectively. The loop corrections (c) and (d) are of order $\hbar^2 n^2$ and are down by a factor of n^2 compared to (a) and (b).

and suppose that $\hbar$ is small. The integral is then dominated by a saddle point of the action: $S'(\phi_{\rm cl})=0$. The semiclassical expansion is generated by shifting $\phi=\phi_{\rm cl}+\sqrt{\hbar}\xi$ and expanding in ξ:

$$\langle\phi^n\rangle=\phi_{\rm cl}^n+\frac{n(n-1)}{2!}\hbar\langle\xi^2\rangle\phi_{\rm cl}^{n-2}+\frac{n(n-1)(n-2)(n-3)}{4!}\hbar^2\langle\xi^4\rangle\phi_{\rm cl}^{n-4}+\cdots. \quad (5)$$

It is clear that the semiclassical expansion breaks down if n is sufficiently large, because the expansion parameter is not really $\hbar$, but $\hbar n^2$. The semiclassical series can be easily resummed in the double-scaling limit when $\hbar$ goes to zero and n goes to infinity with $n\sqrt{\hbar}$ held fixed. Simplifications occur because loops are suppressed by extra powers of $\hbar$ and only tree diagrams survive the double-scaling limit (Fig. 1). It is easy to show that the relevant tree diagrams exponentiate:

$$\langle(\phi_{\rm cl}+\sqrt{\hbar}\xi)^n\rangle=\phi_{\rm cl}^n\left\langle\left(1+\frac{\sqrt{\hbar n^2}}{\phi_{\rm cl}n}\xi\right)^n\right\rangle \to\phi_{\rm cl}^n\left\langle\exp\left(\frac{\sqrt{\hbar n^2}}{\phi_{\rm cl}}\xi\right)\right\rangle_{\rm Gauss}. \quad (6)$$

The Gaussian average can be easily computed:

$$\langle\phi^n\rangle\approx\phi_{\rm cl}^n\exp\left(\frac{\hbar n^2}{2\phi_{\rm cl}^2}D\right), \quad (7)$$

where $D=[S''(\phi_{\rm cl})]^{-1}$ is the propagator of ξ. To facilitate the double-scaling limit, it is convenient to set $\phi=\phi_{\rm cl}e^{\zeta}$ from the start and to treat ζ as a Gaussian fluctuation.

The parameter of the semiclassical expansion in AdS string theory is $1/\sqrt{\lambda}$, which plays the role of the sigma-model coupling α' and is analogous to $\hbar$ of the toy model. The counterpart of n is the R charge J. The toy model suggests that $J^2/\sqrt{\lambda}$ is an expansion parameter of the α' expansion and that the expansion breaks down at $J\sim\lambda^{1/4}$.

Let us now expand the exact OPE coefficient (3) in powers of $1/\sqrt{\lambda}$:

$$c_J(\lambda)=\frac{1}{2}\sqrt{J\lambda}\left(1-\frac{J^2-1}{2\sqrt{\lambda}}+\frac{J^4-4J^2+3}{8\lambda}+\cdots\right). \quad (8)$$

This has precisely the same form as Eq. (5) and, indeed, $J^2/\sqrt{\lambda}$ emerges as a parameter of the strong-coupling expansion. Moreover, the series exponentiate in the double-scaling limit of large J and large λ:[1]

$$c_J(\lambda)\approx\frac{1}{2}\sqrt{J\lambda}\exp\left(-\frac{J^2}{2\sqrt{\lambda}}\right). \quad (9)$$

These observations suggest that the large-J limit of OPE is described by the double-scaling limit of the α' expansion in the $AdS_5\times S^5$ sigma model.

There is another way to take the limit of large n in the toy model. Its counterpart in string theory closely follows Polyakov's approach [20] to string amplitudes with large angular momentum.[2] The idea is to treat an operator insertion ϕ^n on an equal footing with the action, that is, to minimize $W(\phi)=S(\phi)-\hbar n\ln\phi$ instead of $S(\phi)$. The expectation value then takes the form $\langle\phi^n\rangle\approx\exp(-f(\hbar n)/\hbar)$, which clearly assumes that $n\sim\hbar^{-1}$. In the AdS/CFT context, this translates into the Berenstein-Maldacena-Nastase (BMN) scaling $J\sim\sqrt{\lambda}$, which leads to the plane wave limit of $AdS_5\times S^5$ geometry in the closed-string sector [2]. Does the BMN limit make sense for the OPE coefficients (3)? A simple calculation (see Appendix A) shows that it does and that OPE coefficients indeed have the expected exponential form in the limit:

$$c_J\propto\exp\{-\sqrt{\lambda}[1-\sqrt{j^2+1}-j\ln(\sqrt{j^2+1}-j)]\}, \quad (10)$$

where $j=J/\sqrt{\lambda}$. When j goes to zero, this expression reduces to Eq. (9). Again, there is strong evidence that the BMN limit of OPE coefficients can be computed in the sigma model by semiclassical techniques.

III. OPE EXPANSION AT LARGE J

The Wilson loop operator which couples to strings in $AdS_5\times S^5$ is defined as [8]

$$W(C)=\frac{1}{N}\operatorname{tr}\mathrm{P}\exp\left[\oint_C ds[iA_\mu(x)\dot{x}^\mu+\Phi_i(x)\theta^i|\dot{x}|]\right], \quad (11)$$

where θ_i is a unit six-vector: $\theta^2=1$. The OPE coefficients of this operator can be extracted from the normalized two-point function

$$\frac{\langle W({\rm circle})\mathcal{O}_J(x)\rangle}{\langle W({\rm circle})\rangle}=\frac{1}{N}c_J(\lambda)\frac{1}{|x|^{2J}}2^{J/2}\mathbf{Y}_J(\theta)+\cdots, \quad (12)$$

where omitted terms correspond to descendants of $\mathcal{O}_J$ and are suppressed by powers of $1/|x|$. A factor of $2^{J/2}$ is introduced for later convenience and $\mathbf{Y}_J(\theta)$ is the spherical function associated with the operator $\mathcal{O}_J$:

[1] A derivation is given in Appendix A.
[2] I would like to thank A. Tseytlin for this remark.

$$\mathbf{Y}_J(\theta)=\left(\frac{\theta_1+i\theta_2}{\sqrt{2}}\right)^J. \tag{13}$$

A convenient choice is $\theta^i=(1,0,\ldots,0)$, since then

$$2^{J/2}\mathbf{Y}_J(1,0,\ldots,0)=1. \tag{14}$$

For the two-point function (12), the free-field approximation is believed to be exact and gives the result (3) quoted in the Introduction.

On the string side of the AdS/CFT correspondence, Wilson loops are associated with open strings that end on the boundary of $AdS_5\times S^5$. The expectation value of a Wilson loop is equal to the sigma-model partition function with boundary conditions set by the contour C and by the point θ_i in S^5:

$$\langle W(C)\rangle=\int DX^\mu DYD\Theta_i Dh_{ab}D\vartheta^\alpha\delta(\Theta^2-1)\times\exp\Bigg[-\frac{\sqrt{\lambda}}{4\pi}\int_D d^2\sigma\sqrt{h}h^{ab}\times\left(\frac{\partial_aX^\mu\partial_bX^\mu+\partial_aY\partial_bY}{Y^2}+\partial_a\Theta_i\partial_b\Theta_i\right)+\text{fermions}\Bigg]. \tag{15}$$

Here, ϑ^α are world-sheet fermions and y is the AdS scale.[3] The boundary of $AdS_5\times S^5$ is at $y=0$.

The chiral primary operators of R charge J are dual to the Jth Kaluza-Klein (KK) modes on S^5 of a particular supergravity field. The two-point function (12) describes emission of one such mode by a source at point x on the boundary of AdS_5, its propagation in the bulk, and subsequent absorption by the string world-sheet. When $|x|$ is large, the supergraviton propagator associated with a dimension-J operator behaves as $y^J/|x|^{2J}$. The denominator gives rise to an overall factor of $1/|x|^{2J}$ in the correlation function, and Y^J can be regarded as a part of the vertex operator which couples the supergravity mode to the world-sheet. The OPE coefficient is essentially a one-point function of this vertex operator. In other words, it is an overlap of the boundary state associated with the Wilson loop with the graviton state associated with the local operator. It is easy to guess which terms in the vertex operator are responsible for enhancement of the α' expansion at large J. Those are the factor of Y^J, which comes from the supergraviton propagator, and $\mathbf{Y}_J(\Theta)$, which is the wave function of the Jth KK mode on S^5. These simple arguments determine the vertex operator up to a coefficient. A more precise form of the vertex operator, including all coefficients, was derived in [13]:

$$c_J=2^{J/2}\sqrt{J\lambda}\,\frac{J+1}{4\pi}\left\langle\int d^2\sigma\sqrt{h}Y^{J+2}\mathbf{Y}_J(\Theta)\right\rangle. \tag{16}$$

The averaging over string fluctuations here is determined by the partition function (15). The above expression still contains some approximations [13]: an exact vertex operator is more complicated; in particular, it contains world-sheet derivatives and fermions, but those appear only in the sigma-model loops, so Eq. (16) is sufficient for semiclassical calculations.

It is convenient to parametrize the five-sphere by two angles ψ and φ and a unit three-vector $\mathbf{n}$:

$$\theta=(\cos\psi\cos\varphi,\cos\psi\sin\varphi,\sin\psi\mathbf{n}),\quad \mathbf{n}\in S^3. \tag{17}$$

In this parametrization, the spherical functions read

$$\mathbf{Y}_J(\psi,\varphi,\mathbf{n})=2^{-J/2}(\cos\psi)^J e^{iJ\varphi}, \tag{18}$$

and the OPE coefficients become

$$c_J=\sqrt{J\lambda}\,\frac{J+1}{4\pi}\left\langle\int d^2\sigma\sqrt{h}Y^{J+2}(\cos\Psi)^J e^{iJ\Phi}\right\rangle. \tag{19}$$

At large λ, the path integral is dominated by a saddle point, a minimal surface with a fixed boundary. For a circle of unit radius, the classical solution is [13,14]

$$Y_{\rm cl}=z,\quad X^1_{\rm cl}=r\cos\phi,\quad X^2_{\rm cl}=r\sin\phi,$$

$$\Psi_{\rm cl}=0,\quad \Phi_{\rm cl}=0,\quad r=\sqrt{1-z^2}. \tag{20}$$

In the coordinates (z,ϕ), the induced metric takes the form

$$g_{ab}d\sigma^a d\sigma^b=\frac{1}{r^2z^2}dz^2+\frac{r^2}{z^2}d\phi^2. \tag{21}$$

Substitution of the classical solution into the expectation value (19) gives $c_J=\sqrt{J\lambda}/2$ [13], which coincides with the strong-coupling limit of the OPE coefficient computed in SYM theory (3). This result can be improved by taking into account string fluctuations. In the scaling limit $\lambda\to\infty$, $J\to\infty$, $J^2/\sqrt{\lambda}$ fixed, we can repeat the same steps as in the toy model of the previous section, since the expectation value we need to calculate has essentially the same structure as Eq. (4).

The first step is to expand the action of the sigma model around the classical solution. We only need the quadratic part of the action, but even that is not very simple, because various fluctuations mix. The diagonalization and expansion of fluctuations in normal modes is a nontrivial problem. For the circular loop, it was solved in Sec. 5 of [15]. The results that will be necessary to calculate the OPE coefficients are briefly summarized below. The metric fluctuations are eliminated by imposing a background conformal gauge: $h_{ab}=e^\chi g_{ab}$,

[3] In what follows, the bosonic world-sheet coordinates are denoted by capital letters: Y, X^μ, etc.

where χ is an arbitrary conformal factor. In the notations of [15], the world-sheet fields are expanded around the classical solution as follows:

$$\begin{aligned} Y &= ze^{\zeta^4}, \\ X^1 &= (r+z\zeta^1)\cos\phi - z\zeta^0\sin\phi, \\ X^2 &= (r+z\zeta^1)\sin\phi + z\zeta^0\sin\phi. \end{aligned} \tag{22}$$

We should treat ζ^a and all other fields, including Φ and Ψ, as Gaussian fluctuations. The radial fluctuations ζ^1,ζ^4 mix and are expanded in normal modes as

$$\zeta^4 = z\tilde{\zeta}^4 - r\tilde{\zeta}^1, \tag{23}$$

$$\zeta^1 = r\tilde{\zeta}^1 + z\tilde{\zeta}^4. \tag{24}$$

The relevant part of the action for normal modes is very simple [15]:

$$S = \frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma\sqrt{g}g^{ab}[\partial_a\Phi\partial_b\Phi + \partial_a\tilde{\zeta}^4\partial_b\tilde{\zeta}^4 + 2(\tilde{\zeta}^4)^2]. \tag{25}$$

Substituting Eq. (22) into the correlation function (19), we get:

$$c_J = \sqrt{J\lambda}\,\frac{J+1}{4\pi}\int_0^{2\pi}d\phi\int_0^1 dzz^J\langle e^{J(\zeta^4+i\Phi)}\rangle_{\rm Gauss}. \tag{26}$$

The fluctuations of Ψ do not contribute to the leading order in J, because expansion of $(\cos\Psi)^J$ in Ψ starts from the quadratic term and leads to an $O(J/\sqrt{\lambda})$ correction to the OPE coefficient, which vanishes in the scaling limit we consider. The Gaussian average is defined by the action (25) and is expressed in terms of standard scalar propagators on the minimal surface:

$$\left(-\frac{1}{\sqrt{g}}\partial_a g^{ab}\sqrt{g}\partial_b + m^2\right)G_{m^2}(\sigma,\sigma') = \frac{1}{\sqrt{g}}\delta(\sigma,\sigma'). \tag{27}$$

Doing the Gaussian integration, we get

$$\langle e^{J(\zeta^4(\sigma)+i\Phi(\sigma))}\rangle = \lim_{\sigma'\to\sigma}\exp\left[\frac{\pi J^2}{\sqrt{\lambda}}[z^2G_2(\sigma,\sigma') + r^2\tilde{G}(\sigma,\sigma') - G_0(\sigma,\sigma')]\right], \tag{28}$$

where $\tilde{G}(\sigma,\sigma')$ is the propagator of $\tilde{\zeta}^1$, which will drop out at the end of the calculation. Individual propagators are singular at coinciding points, but the singularities cancel between the AdS_5 and the S^5 contributions, so the one-point function of the vertex operator is finite. It remains to integrate it over the world-sheet. The integration is, in fact, trivial because

$$(J+1)\int_0^1 dzz^J f(z) = f(1) + O(1/J) \tag{29}$$

for any smooth function. Because of the z^J factor, the main contribution comes from the point on the world-sheet that is farthest from the boundary of AdS_5. Since $r\equiv\sqrt{1-z^2}=0$ at $z=1$, the propagator $\tilde{G}(\sigma,\sigma')$ drops out from the final result:

$$c_J = \frac{1}{2}\sqrt{J\lambda}\exp\left(-\frac{A\pi J^2}{\sqrt{\lambda}}\right), \tag{30}$$

where

$$A = \lim_{\sigma'\to\sigma}[G_0(\sigma,\sigma') - G_2(\sigma,\sigma')]. \tag{31}$$

Agreement with the SYM prediction (9) requires $A=1/2\pi$. A simple calculation, which is deferred to Appendix B, gives precisely the right value for A.

Although calculations have been done only for the circle, an enhancement of the semiclassical expansion at $J\sim\lambda^{1/4}$ must be a generic feature. It is not hard to see that OPE coefficients of an arbitrary Wilson loop behave as

$$a_J\sqrt{\lambda}\exp\left(-\frac{A\pi J^2}{\sqrt{\lambda}}\right)$$

at large J and large λ. The constants a_J and A depend on the shape of the loop and on a particular choice of local operators. A must be positive, because operators of large R charge must be exponentially suppressed in the OPE expansion, according to quite general arguments [10,21].

IV. BMN LIMIT OF OPE COEFFICIENTS

When $J\sim\sqrt{\lambda}$, the exponent in the vertex operator and the sigma-model action are of the same order. The integral over string world-sheets is then dominated by a solution of classical equations of motion with a source term that comes from the operator insertion [20]. It is convenient to use an exponential parametrization of the AdS radial coordinate: $y=e^{-p}$. Then the string action takes the following form in the conformal gauge:

$$S = \frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma'[(\partial P)^2 + e^{2P}(\partial X^\mu)^2 + (\partial\Psi)^2 + \cos^2\Psi(\partial\Phi)^2] + JP(\sigma) - iJ\Phi(\sigma) - J\ln\cos\Psi(\sigma). \tag{32}$$

Not only the path integral over string world-sheets, but also the integral over the position of operator insertion, is semiclassical, so the action should be minimized with respect to the point of operator insertion σ, too.

Before going to the general case, let us first consider the solution of the equations of motion at $J=0$. To find it, we should transform Eq. (20) to the conformal gauge:

$$Y_{\rm cl}=\tanh\tau,\quad X^1_{\rm cl}=\frac{\cos\phi}{\cosh\tau},\quad X^2_{\rm cl}=\frac{\sin\phi}{\cosh\tau}. \tag{33}$$

The induced metric on the minimal surface is conformal to the unit metric in the (τ,ϕ) coordinates: $ds^2=(d\tau^2+d\phi^2)/\sinh^2\tau$. The world-sheet is a semi-infinite cylinder, which is mapped to the hemisphere in the target space. The center of symmetry on the hemisphere corresponds to $\tau=\infty$.

Returning to the general case of an arbitrary angular momentum, we can see that the minimum of the action in the position of operator insertion is reached at the most symmetric point, when the vertex operator is inserted at $\tau=\infty$. Consequently, the minimal surface possesses rotational symmetry at any J, not only at $J\neq 0$, and we can take the following ansatz for the classical string world-sheet:

$$Y=e^{-P(\tau)},\quad X^1=R(\tau)\cos\phi,$$
$$X^2=R(\tau)\sin\phi,\quad \Phi=\Phi(\tau). \tag{34}$$

The azimuthal angle on S^5 can be put to zero, because $\Psi=0$ satisfies equations of motion even in the presence of the source. Substituting the ansatz into the action, we get

$$S=\sqrt{\lambda}\left\{\frac{1}{2}\int_0^\infty d\tau[\dot P^2+e^{2P}(\dot R^2+R^2)+\dot\Phi^2]\right.$$
$$\left.+jP(\infty)-ij\Phi(\infty)\right\}, \tag{35}$$

where the overdot denotes the derivative in τ. The source term in the action sets boundary conditions at infinity. To understand what exactly these boundary conditions are, it is convenient to map the cylinder to a disk by changing coordinates from τ to $\rho=e^{-\tau}$. The world-sheet metric becomes $ds^2\propto(d\rho^2+\rho^2d\phi^2)$. If a free field with the action normalized as in Eq. (32) interacts with a source of strength J localized at $\rho=0$, it asymptotes $-j\ln\rho$ at the center of the disk. The logarithmic singularity on the disk translates into linear ($\sim j\tau$) growth at infinity on the cylinder. Therefore, the boundary conditions are

$$R(\tau)\to 0,\quad P(\tau)\to -j\tau,\quad \Phi(\tau)\to ij\tau$$
$$(\text{at}\quad \tau\to\infty). \tag{36}$$

Boundary conditions at zero are set by the Wilson loop:

$$R(0)=1,\quad P(0)=\infty,\quad \Phi(0)=0. \tag{37}$$

The solution of equations of motion for Φ is very simple:

$$\Phi=ij\tau. \tag{38}$$

It is complex, but becomes real after Wick rotation on the world-sheet, when the internal metric has the Minkowski signature. The solution then describes the rotation along the big circle of S^5 with constant angular momentum J.

The AdS part of the minimal surface is more complicated. The equations of motion which follow from the action (35) are

$$\ddot P-e^{2P}(\dot R^2+R^2)=0, \tag{39}$$

$$\ddot R+2\dot P\dot R-R=0. \tag{40}$$

These equations admit two first integrals: the energy associated with translations in τ and the dilatation charge associated with the invariance of the action under rescaling of AdS coordinates: $R\to e^{\omega}R,\ P\to P-\omega$. The boundary conditions fix the values of the conserved charges:

$$\dot P^2+e^{2P}(\dot R^2-R^2)=j^2, \tag{41}$$

$$e^{2P}R\dot R-\dot P=j. \tag{42}$$

The first of these two equations is equivalent to Virasoro constraints: it ensures that the induced metric is conformal to the flat metric. The fact that Virasoro constraints follow from the equations of motion and boundary conditions is a consequence of marginality of the vertex operator.

The equations of motion can be easily integrated:

$$Y=e^{-P}=e^{j\tau}[\sqrt{j^2+1}\tanh(\sqrt{j^2+1}\,\tau+\xi)-j],$$
$$R=\frac{\sqrt{j^2+1}\,e^{j\tau}}{\cosh(\sqrt{j^2+1}\,\tau+\xi)}, \tag{43}$$

where

$$\xi=\ln(\sqrt{j^2+1}+j). \tag{44}$$

The solution describes a surface of revolution in AdS_5, which terminates on the boundary and which has an infinite spike that goes down to the horizon. The spike describes emission of the external supergravity state.

There are several subtleties in evaluating the action on the classical solution (43). The action potentially diverges at both limits of integration. The UV divergence at $\tau=\infty$ cancels between the AdS_5 and the S^5 contributions. Here again, the marginality of the vertex operator plays a crucial role. In particular, the boundary terms are finite:

$$\lim_{\tau\to\infty}[jP(\tau)-ij\Phi(\tau)]=-j\ln(\sqrt{j^2+1}-j). \tag{45}$$

The action also diverges at $\tau=0$. This divergence is associated with the singularity of the metric at the boundary of AdS_5. Such divergences are well known and are well understood [8,14]. The correct way to deal with them is very simple: the divergence should be regularized and then subtracted. In the present case, we do not even need to do the subtraction, because the divergence cancels against the normalization factor, which is given by the classical action at $j=0$. Intermediate calculations still require a regularization, which can be implemented by imposing a lower bound on the integration variable τ at some small but finite ε.

Using the conservation of energy (41), we get for the bulk part of the action

$$S_{\rm bulk}=\sqrt{\lambda}\,W(j) \tag{46}$$

with

$$W(j)=\frac{1}{2}\int_{\varepsilon}^{\infty}d\tau(j^2+2e^{2P}R^2+\dot{\Phi}^2)=\int_{\varepsilon}^{\infty}d\tau\frac{R^2}{Y^2}. \tag{47}$$

The integrand here is a total derivative. The easiest way to see it is to introduce

$$B\equiv-\frac{\dot{R}}{R}=\sqrt{j^2+1}\tanh(\sqrt{j^2+1}\tau+\xi)-j. \tag{48}$$

The equations of motion imply that

$$\frac{R^2}{Y^2}=\frac{\dot{B}}{B^2}.$$

So

$$W(j)=\frac{1}{B(\varepsilon)}-\frac{1}{B(\infty)}.$$

Since

$$B(\varepsilon)=\varepsilon-j\varepsilon^2+\cdots$$

and

$$B(\infty)=\sqrt{j^2+1}-j,$$

we get

$$W(j)-W(0)=1-\sqrt{j^2+1}. \tag{49}$$

Collecting together all contributions, Eqs. (45), (46), and (49), we obtain

$$S_{\rm cl}=\sqrt{\lambda}[1-\sqrt{j^2+1}-j\ln(\sqrt{j^2+1}-j)]. \tag{50}$$

The OPE coefficients are determined by the classical action up to an overall factor of order 1:

$$c_J=\text{const}\cdot\exp(-S_{\rm cl}). \tag{51}$$

This coincides with the field-theory prediction (10).

V. PERTURBATIVE REGIME

The two scaling limits discussed in the previous sections are related—the large-J limit with $J\sim\lambda^{1/4}$ can be reproduced from the BMN limit $J\sim\sqrt{\lambda}$, by taking $j\equiv J/\sqrt{\lambda}$ small. What happens in the opposite regime of large j? If j were the only parameter in the problem, increasing j could be achieved either by raising J or by lowering λ. In the latter case, the limit of large j would correspond to the weak-coupling regime, which could be confronted with SYM perturbation theory. Comparison with perturbation theory makes sense if large angular momentum alone can suppress quantum fluctuations of the world-sheet, without any assumptions about the string tension. Otherwise, λ should still be large for the semiclassical approximation to work, and then comparison to perturbation theory makes no sense. The perturbative regime seems to be within the reach of the semiclassical approximation in the closed string sector [2–7]. The analysis below gives some indications that this is true for open strings as well.

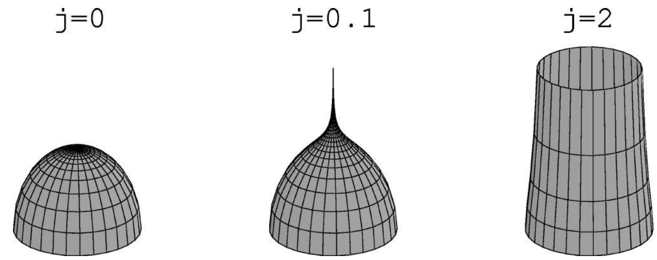

FIG. 2. The classical string world-sheet (43) for different values of angular momentum.

The shape of the classical world sheet (43) strongly depends on the angular momentum (Fig. 2). At small j, the world-sheet deviates very little from the solution without an operator insertion, except in the close vicinity of the origin, where the world-sheet degenerates into an infinite, narrow tube that describes emission of the supergravity mode. The width of the tube grows with j, and the world-sheet becomes almost a perfect cylinder at large j. Indeed, the scale of variation of R in Eq. (43) is j at $j\to\infty$, while the scale of variation of Y is $1/j$, so R is almost constant compared to Y. This, in fact, is a general property which is valid for any Wilson loop, not only for the circle.

For an arbitrary Wilson loop $W(C)$, the minimal surface can be parametrized by the natural parameter s on contour C [such that $(dx^\mu/ds)^2=1$] and the "time" variable τ. An asymptotic solution at large j is described by the following ansatz:

$$X^\mu=x^\mu(s),\quad Y=e^{-P(\tau)},\quad \Phi=ij\tau. \tag{52}$$

The (approximate) time independence of X^μ is consistent with the equations of motion in the sigma model:

$$\frac{\partial^2X^\mu}{\partial s^2}+2\frac{\partial P}{\partial\tau}\frac{\partial X^\mu}{\partial\tau}+\frac{\partial^2X^\mu}{\partial\tau^2}=0. \tag{53}$$

The first term is of order 1 at $j\to\infty$. Assuming that $\partial P/\partial\tau=O(j)$, we find that $\partial X^\mu/\partial\tau=O(1/j)$ and that the last term is negligible. The time dependence of the AdS radius is determined by the equation

$$\frac{\partial^2P}{\partial\tau^2}-e^{2P}=0. \tag{54}$$

Imposing the boundary conditions we get

$$Y=\frac{\sinh(j\tau)}{j}. \tag{55}$$

This solution is universal. It is the same for any Wilson loop. The classical action can be readily computed:

$$S_{\rm bulk}=\sqrt{\lambda}\int_{\varepsilon}^{\infty}\frac{d\tau}{Y^2}=-\sqrt{\lambda}j+\sqrt{\lambda}\,\frac{1}{\varepsilon}\to -J. \tag{56}$$

The $1/\varepsilon$ divergence is spurious and should be subtracted [8,14]. We should also take into account the vertex operator evaluated on the classical solution: a factor of

$$(Y(\infty))^J e^{-iJ\Phi(\infty)}=(1/j)^J.$$

We get for the correlator

$$\langle W(C)\mathcal{O}_J(x)\rangle\propto\left(\frac{\sqrt{\lambda}}{J}\right)^J e^J\propto\frac{\lambda^{J/2}}{J!} \tag{57}$$

for any smooth contour C.

It is not hard to recognize the leading order of SYM perturbation theory in Eq. (57). Indeed, the Wilson loop operator (11) should be expanded to Jth order in scalar fields to get a nonvanishing correlator with $\mathcal{O}_J$. The Wilson loop is an exponential; hence a factor of $1/J!$ will arise. The lowest-order diagram of perturbation theory contains J scalar propagators that connect the Wilson loop with the local operator. Each propagator gives a factor of λ. The operator itself is proportional to $\lambda^{-J/2}$, so the correlator $\langle W(C)\mathcal{O}_J(x)\rangle$ is proportional to $\lambda^{J/2}$ to the first nonvanishing order in perturbation theory.

VI. REMARKS

The OPE coefficients of the circular Wilson loop can be computed to all orders in SYM perturbation theory and, for large R charges, to all orders in the α' expansion in AdS string theory. The results of these calculations completely agree in two scaling limits, $J\sim\lambda^{1/4}$ and $J\sim\lambda^{1/2}$. For the OPE coefficients, the interpolation between the weak-coupling and the strong-coupling regimes can be traced on both sides of the AdS/CFT correspondence.

Most interestingly, the perturbative SYM regime seems to be accessible in string theory. The leading order of perturbation theory is described by an approximate classical solution in the AdS sigma model described in Sec. V. In principle, the equations of motion can be solved by iterations starting from the solution of Sec. V as the zeroth-order approximation. It would be very interesting to understand if the iterative solution of the sigma model is equivalent to ordinary planar perturbation theory. If true, this opens an intriguing possibility for understanding how planar Feynman diagrams arise in AdS string theory.

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APPENDIX A: THE LARGE-J LIMIT IN FIELD THEORY

The following integral representation of the modified Bessel functions is useful in taking the large-J limit:

$$I_J(\sqrt{\lambda})=\frac{(\sqrt{\lambda}/2)^J}{\sqrt{\pi}\,\Gamma\left(J+\frac{1}{2}\right)}\int_{-1}^{1}dx(1-x^2)^{J-1/2}e^{\sqrt{\lambda}x}. \tag{A1}$$

When $\lambda\to\infty$, $J\to\infty$, and $J^2/\sqrt{\lambda}$ is fixed, a convenient change of integration variable is $t=\sqrt{\lambda}(x+1)$:

$$I_J(\sqrt{\lambda})=\frac{\lambda^{-1/4}e^{\sqrt{\lambda}}}{\sqrt{2\pi}\,\Gamma\left(J+\frac{1}{2}\right)}\int_0^{2\sqrt{\lambda}}dt\,e^{-t}t^{J-1/2}\left(1-\frac{t}{2\sqrt{\lambda}}\right)^{J-1/2}. \tag{A2}$$

The strong-coupling expansion is generated by expanding the last factor in $t/2\sqrt{\lambda}$. At large J, the integral is dominated by $t\sim J$, and the expansion breaks down. Instead of expanding the last factor, we should replace it by an exponential:

$$\left(1-\frac{t}{2\sqrt{\lambda}}\right)^{J-1/2}=\left(1-\frac{t(J-1/2)}{2\sqrt{\lambda}(J-1/2)}\right)^{J-1/2}\approx\exp\left[-\frac{t(J-1/2)}{2\sqrt{\lambda}}\right].$$

The integration over t is then trivial:

$$I_J(\sqrt{\lambda})\approx\frac{\lambda^{-1/4}e^{\sqrt{\lambda}}}{\sqrt{2\pi}}\left(1-\frac{J-1/2}{2\sqrt{\lambda}}\right)^{-(J-1/2)}\approx\frac{\lambda^{-1/4}e^{\sqrt{\lambda}}}{\sqrt{2\pi}}\exp\left(-\frac{J^2}{2\sqrt{\lambda}}\right).$$

Dividing by the large-λ asymptotics of $I_1(\sqrt{\lambda})$,

$$I_1(\sqrt{\lambda})\approx\frac{\lambda^{-1/4}e^{\sqrt{\lambda}}}{\sqrt{2\pi}},$$

we get Eq. (9).

In the BMN limit $\lambda\to\infty$, $J=j\sqrt{\lambda}$, and j is finite. The integral (A1) can be computed by the saddle-point method in this limit. The integral is dominated by the maximum of

$$\mathcal{S}(x)=\sqrt{\lambda}[x+j\ln(1-x^2)],$$

which is reached at $x\equiv x_0=\sqrt{j^2+1}-j$, and

$$\mathcal{S}(x_0)=\sqrt{\lambda}\{\sqrt{j^2+1}-j+\ln[2j(\sqrt{j^2+1}-j)]\}.$$

The overall factor in Eq. (A1) also has a well-defined BMN limit:

$$\frac{(\lambda/2)^J}{\Gamma\left(J+\frac{1}{2}\right)}\approx\text{const}\cdot\exp[\sqrt{\lambda}j(1-\ln 2j)].$$

Consequently,

$$I_J(\sqrt{\lambda})\propto\exp\{\sqrt{\lambda}[\sqrt{j^2+1}+j\ln(\sqrt{j^2+1}-j)]\}. \quad \text{(A3)}$$

Dividing by the normalization factor $I_1(\sqrt{\lambda})$, we get Eq. (10).

APPENDIX B: GREEN'S FUNCTIONS

The classical string world-sheet for the circular loop has a geometry of AdS_2. The metric (21) can be put into a more familiar form by an inversion:

$$x^\mu\to\frac{(x+x_0)^\mu}{(x+x_0)^2+z^2},\quad z\to\frac{z}{(x+x_0)^2+z^2}, \quad \text{(B1)}$$

with $x_0=(1,0,0,0)$. This transformations maps a hemisphere (20) onto a half plane $x^\mu=(1/2,u,0,0)$, $-\infty<u<\infty$, $0<z<\infty$, which is an AdS_2 subspace of AdS_5 with the metric $ds^2=(dz^2+du^2)/z^2$.

The scalar propagators in AdS_2 are well known [22–24]. The general result for a scalar field in $AdS_{(d+1)}$ with $m^2=\Delta(\Delta-d)$ is [25]

$$G_{m^2}(z,u;z',u')=\frac{\Gamma(\Delta)}{2^\Delta\pi^{d/2}\Gamma(\Delta-d/2)(2\Delta-d)}\xi^\Delta \times F\left(\frac{\Delta}{2},\frac{\Delta+1}{2};\Delta-\frac{d}{2}+1;\xi^2\right), \quad \text{(B2)}$$

where

$$\xi=\frac{2zz'}{z^2+z'^2+(u-u')^2}. \quad \text{(B3)}$$

Specifying to $d=1$ and to $m^2=0,2$, we get

$$G_0(\xi)=\frac{1}{4\pi}\ln\left(\frac{1+\xi}{1-\xi}\right)(\Delta=1),$$

$$G_2(\xi)=\frac{1}{4\pi\xi}\ln\left(\frac{1+\xi}{1-\xi}\right)-\frac{1}{2\pi}\ (\Delta=2). \quad \text{(B4)}$$

In the limit of coinciding points, $\xi\to1$. Therefore,

$$A=\lim_{\xi\to1}[G_0(\xi)-G_2(\xi)]=\frac{1}{2\pi}. \quad \text{(B5)}$$

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